Full averaging of set integrodifferential equations

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As is generally known the absence of exact universal research methods for many important problems of analytical dynamics has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms. The averaging methods combined with the asymptotic representations began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. It became possible due to the works of N.M. Krylov, N.N. Bogolyubov, Yu.A. Mitropolskij, V.M. Volosov, N.N. Moiseev, etc. (see [1-8]).

In recent years the development of the calculus in metric spaces has attracted some attention [6-12]. Earlier, F.S. de Blasi, F. Iervolino [13] started the investigation of set differential equations (SDEs) in semilinear metric spaces. This has now evolved into the theory of SDEs as an independent discipline: properties of solutions [6-45], the impulse equations [6,7,41], control systems [42-45] and asymptotic methods [6-8,46-50]. On the other hand, SDEs are useful in other areas of mathematics. For example, SDEs are used, as an auxiliary tool, to prove existence results for differential inclusions [6,30,35,39]. Also, one can employ SDEs in the investigation of fuzzy differential equations [7,10,25-27, 29,30]. Moreover, SDEs are a natural generalization of the usual ordinary differential equations in finite (or infinite) dimensional Banach spaces.

In this talk we prove the substantiation of the method of full averaging for the set integrodifferential equations with small parameter. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

1. Preliminaries

Let $comp\left(R^n\right)\left(conv\left(R^n\right)\right)$ be a set of all nonempty (convex) compact subsets from the space R^n,

$$h\left(A,B\right)=\min_{r\geq 0}\left\{\ S_r\left(A\right)\supset B,\ S_r\left(B\right)\supset A\right\}$$

be Hausdorff distance between sets A and B, $S_r\left(A\right)$ is r-neighborhood of set A.

Let A, B, C be in $conv\left(R^n\right)$. The set C is the Hukuhara difference of A and B, if $B+C=A$, i.e. $C=A\overset{H}{-}B$.

From Radstrom's Cancellation Lemma [51], it follows that if this difference exists, then it is unique.

Definition [52]. A mapping $X:[0,T]\to conv\left(R^n\right)$ is differentiable in the sense of Hukuhara at $t\in[0,T]$ if for some $\delta>0$ the Hukuhara differences

$$X\left(t+\Delta\right)\overset{H}{-}X\left(t\right),\ X\left(t\right)\overset{H}{-}X\left(t-\Delta\right)$$

exists in $conv\left(R^n\right)$ for all $0<\Delta<\delta$ and there exists an $DX\left(t\right)\in conv\left(R^n\right)$ such that

$$\lim_{\Delta\to 0_+} h(\frac{1}{\Delta}(X\left(t+\Delta\right)\overset{H}{-}X\left(t\right)),DX(t))=0$$

and

$$\lim_{\Delta\to 0_+} h(\frac{1}{\Delta}(X\left(t\right)\overset{H}{-}X\left(t-\Delta\right)),DX(t))=0.$$

Here $DX\left(t\right)$ is called the Hukuhara derivative of $X\left(t\right)$ at t.

2. The scheme of full average

Consider the Cauchy problem with small parameter

$$DX = \varepsilon F(t, X, \int_0^t \Phi\left(t, s, X\left(s\right)\right)ds), \qquad X\left(0\right) = X_0, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, $t, s \in R_+$, $F : R_+ \times conv\left(R^n\right) \times conv\left(R^m\right) \to conv\left(R^n\right)$ is a set mapping, $\Phi : R_+ \times R_+ \times conv\left(R^n\right) \to comp\left(R^m\right)$ is a set mapping, Here the integral is understood in the sense of [52] (the integral exists for example if $X\left(\cdot\right)$ is measurable and the real mapping $t \to h\left(X\left(t\right), \{0\}\right)$ is integrable on $I \subset R_+$), $X_0 \in conv\left(R^n\right)$.

Definition. A mapping $X : [0, T] \to conv\left(R^n\right)$ is a solution to the problem (1) if and only if it is continuous and satisfies the integral equation

$$X\left(t\right) = X_0 + \varepsilon \int_0^t F(\tau, X\left(\tau\right), \int_0^\tau \Phi\left(\tau, s, X\left(s\right)\right)ds)\, d\tau$$

for all $t \in [0, T]$.

In the beginning this section we associate with the equation (1) the following full averaged integrodifferential equation

$$DY = \varepsilon \overline{F}(Y), \qquad Y(0) = X_0, \tag{2}$$

where

$$\lim_{T\to\infty} h\left(\frac{1}{T}\int_0^T F_1(t, X)\,dt, \overline{F}(X)\right) = 0, \tag{3}$$

$$F_1(t, X) \equiv F(t, X, \Psi(t, X)), \ \Psi(t, X) \equiv \int_0^t \Phi(t, s, X)\,ds. \tag{4}$$

Remark. In the beginning we will consider a case when the limit (3) exists.

Theorem. Let in domain $Q=\{(t,X)\mid t\ge 0,\ X\in G\in conv(R^n)\}$ the following hold:

1) $F(t,X,Z)$ is continuous in $(t,X,Z)\in R_+\times G\times conv(R^m)$;

2) $\Phi(t,s,X)$ is continuous in $(t,s,X)\in R_+\times R_+\times G$;

3) there exist continuous function $\mu(t,s)$, and constant λ such that

$$h(F(t,X_1,Z_1),F(t,X_2,Z_2))\le\lambda[h(X_1,X_2)+h(Z_1,Z_2)],$$

$$h(\Phi(t,s,X_1),\Phi(t,s,X_2))\le\mu(t,s)h(X_1,X_2),$$

for any $X_1,X_2\in G,\ Z_1,Z_2\in conv(R^m)$;

4) $\lim\limits_{t\to\infty}\bar{\mu}_0(t)=0$, where $\bar{\mu}_0(t)\equiv\dfrac{1}{t}\int\limits_0^t\mu_0(\tau)d\tau$, $\mu_0(\tau)\equiv\int\limits_0^\tau\mu(\tau,s)ds$;

5) there exist constants ν, M such that $h(\bar{F}(X_1),\bar{F}(X_2))\le\nu h(X_1,X_2)$, $h(\bar{F}(X),\{0\})\le M$,

for any $X,X_1,X_2\in G$;

6) the limit (3) exist uniformly in $X\in G$;

7) for any $X_0\in G'\subset G$ and $t\ge 0$ the solution of the equation (2) together with a σ-neighborhood belong to the domain G.

Then for any $\eta>0$ and $L>0$ there exists $\varepsilon^0(\eta,L)\in(0,\sigma]$ such that for all $\varepsilon\in(0,\varepsilon^0]$ and $t\in[0,L\varepsilon^{-1}]$ the following statement fulfill: $h(X(t),Y(t))<\eta$, where $X(t)$, $Y(t)$ are the solutions of the initial and the full averaged equations.

Now we consider a case when the limit (3) not exists. Then we associate with the equation (1) the following integrodifferential equation

$$DY = \varepsilon F\left(t, Y, \int_0^t \Phi\left(t, s, Y\left(t\right)\right) ds\right), \qquad Y\left(0\right) = X_0. \tag{5}$$

Theorem. Let in domain $Q=\left\{\left(t,X\right)\mid t\geq 0,\ X\in G\in conv\left(R^{n}\right)\right\}$ the following hold:

1) $F\left(t,X,Z\right)$ is continuous in $\left(t,X,Z\right)\in\ R_{+}\times G\times conv\left(R^{m}\right)$;

2) $\Phi\left(t,s,X\right)$ is continuous in $\left(t,s,X\right)\in R_{+}\times R_{+}\times G$;

3) there exist continuous function $\mu\left(t,s\right)$, and constants $\lambda\ M$ such that

$$h(F\left(t,X_{1},Z_{1}\right),F\left(t,X_{2},Z_{2}\right))\leq\lambda[h\left(X_{1},X_{2}\right)+h\left(Z_{1},Z_{2}\right)],$$

$$h(\Phi\left(t,s,X_{1}\right),\Phi\left(t,s,X_{2}\right))\leq\mu\left(t,s\right)h\left(X_{1},X_{2}\right),\ \ h(F\left(t,X,Z\right),\left\{0\right\})\leq M,$$

for any $X,X_{1},X_{2}\in G,\ Z,Z_{1},Z_{2}\in conv\left(R^{m}\right)$;

4) $\lim\limits_{t\to\infty}\overline{\mu}_{0}\left(t\right)=0$, where $\overline{\mu}_{0}\left(t\right)\equiv\ \frac{1}{t}\ \int\limits_{0}^{t}\mu_{0}\left(\tau\right)d\tau,\ \mu_{0}\left(\tau\right)\equiv\int\limits_{0}^{\tau}\mu\left(\tau,s\right)ds$;

5) for any $X_{0}\in G'\subset G$ and $t\geq 0$ the solutions of the equations (1) and (5) together with a σ-neighborhood belong to the domain G.

Then the following statement fulfill: $h\left(X\left(t\right),Y\left(t\right)\right)<\lambda ML\beta\left(\varepsilon\right)e^{\lambda L+\lambda\beta\left(\varepsilon\right)}$, where $t\in[0,L\varepsilon^{-1}]$, $\beta\left(\varepsilon\right)=\sup\limits_{0\leq\tau\leq L}\tau\overline{\mu}_{0}\left(\frac{\tau}{\varepsilon}\right)$.

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